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Analysis of the successful and blocked events in the $Geo/Geo/c$ retrial queue

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ABSTRACT

This paper considers a discrete-time queue of $Geo/Geo/c$-type with geometric repeated attempts, Artalejo et al. (2008) [18]. We investigate the probability distributions, the first and second moments and the cross moments of the successful and blocked events made by the external and repeated customers. Several numerical examples and a cost function illustrate the analysis.

1. Introduction

The advent of the new technologies has made that the discrete-time queueing theory causes a great stir in the scientific literature on computers and operations research, applied mathematics and related fields. The reason is that the discrete-time systems are more appropriate than their continuous-time counterparts to model computer and telecommunication systems. The discrete-time scale actually reveals the nature of an underlying application: e.g., the clock time unit in a computer system, the fixed size data units (bits, bytes, fixed length packets) on a communication channel,.... That is why many applications of the discrete-time queues fall on the performance analysis of Broadband Integrated Services Digital Network, Asynchronous Transfer Mode and related computer communication technologies, which are not adaptable in the continuous-time models. Interested researchers are referred to the monographs [1–3], where a detailed discussion and applications of discrete-time queues can be found. Additionally, Hunter [4] considered some discrete-time queueing models and Daduna [5] networks of discrete-time queues.

Classical queueing systems are characterized by being the customers in continuous contact with the server, i.e., they can "see" whether or not the server is busy and thus commence their service immediately whenever the service station becomes idle. On the contrary, the customers in a retrial queueing system do not know the server's state and consequently must verify it from time to time. Specifically, in retrial queues (also known as queues with repeated calls, returning customers or repeated orders), the arriving customers who find all servers busy leave the service area temporarily, but repeat their request some random time later. Retrial queues arise to solve problems in telephony and have wide practical applications in call centers, computer and telecommunication networks, stacked aircraft waiting to land, queues of retail shoppers who may leave a long waiting line hoping to return later when the line may be shorter, so that the area of possible applications of such systems is wide [6]. A detailed review of the main mathematical results and bibliographical references on the retrial queues can be found in [7–11]. It should also be asserted that Yang and Li [12] pioneered the study of the discrete-time retrial queues. To show the researchers' interest on the retrial queueing theory in the last decade, we mention some

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recent publications [6,13–23], a special issue on ''Algorithmic Methods in Retrial Queues'' in the journal *Annals of Operations Research*, a special issue on ''Advances in Retrial Queues'' of the *European Journal of Operational Research*, as well as a special issue on ''Algorithmic and Computational Methods in Retrial Queues'' in the journal *Computers and Operations Research*.

The literature about retrial queues is extensive in the sense of researching on a wide variety of different queueing models the following subjects: the embedded Markov chain and its ergodicity condition, the stationary distribution, the busy period, the number of customers served in a busy period, the waiting time,.... However, to the best of our knowledge, Amador and Artalejo [15–17] pioneered in defining and analyzing four new descriptors: the number of successful and blocked retrials and the number of successful and blocked external arrivals during a busy period. Their studies shed light about two indistinguishable streams of arrivals in all retrial queueing systems (external and repeated arrivals), increased the knowledge around the group of customers in orbit and provided information about the difficulty of access to the server. Moreover, they answered questions relative to: how many attempts do the repeated customers make to obtain service? and a customer who has just received its service, is it external or repeated? Specifically, Amador and Artalejo investigated the distribution of these four descriptors for a $M/M/c$ retrial queue [15], for an $M/G/1$ retrial queue [16] and from a transient point of view [17]. Nevertheless, their analysis is restricted to the continuous-time field. That is why, the objective of this paper is to analyze the distribution of these new descriptors for a discrete-time $Geo/Geo/c$ retrial queueing system [18], and so to extend the study of these new performance descriptors to the discrete-time scope. To conclude this paragraph, we would like to point out that the complexity of the discrete-time queueing models increases because the probability that two or more events occur simultaneously at every slot is positive.

The remainder of the paper is structured as follows. The next section describes the queueing system under analysis, introduces its corresponding mathematical model and defines performance descriptors of interest. In Section 3, we find a recursive scheme to obtain the exact distribution of the number of successful retrials. By means of the truncated model with finite orbit, Sections 4 and 5 provide approximate methods for the probability mass functions of the number of blocked retrials and successful arrivals, respectively. For the truncated model, Section 6 presents algorithmic solutions for computing the first moments, the second moments and the cross moments of the number of successful retrials, blocked retrials and successful arrivals. Finally, in Section 7, we show some numerical results about the expectations, the coefficients of variation, the correlation coefficients and the probability distributions of the events in the study. In order to point out some practical applications from the economic point of view, Section 7 concludes establishing and minimizing a cost function.

2. Model description and performance descriptors

We next describe a discrete-time $Geo/Geo/c$ retrial queue. We assume that the time axis is divided into equal intervals of unit length, called slots. Customers arrive at the system according to a Bernoulli process with probability $p \in (0, 1)$, that is, the interarrival times are independent random variables with geometric distribution given by the probability mass function (p.m.f.) $p\bar{p}^{k-1}$, $k \geq 1$, where $\bar{p} = 1 - p$. Service is rendered by $c \geq 1$ identical servers with service times geometrically distributed with p.m.f. $q\bar{q}^{k-1}$, $k \geq 1$, where $\bar{q} = 1 - q$. If a customer arrives and finds some server free, it immediately starts its service and it leaves the system after being served. Otherwise, if all servers are busy, it has to go to a retrial group, called an orbit, in order to repeat its request again after a random amount of time. Customers in the orbit behave independently of each other and retry with probability s at every time slot, i.e., their retrial times are independent geometric random variables with p.m.f. $s\bar{s}^{k-1}$, $k \geq 1$, where $\bar{s} = 1-s$. Moreover, we assume that process of primary arrivals, service times and interretrial times are mutually independent.

We consider a generalized early arrival scheme (G-EAS) in which every event occurs around the slot boundaries. At a given slot boundary t, departures occur in $(t-, t)$ while primary arrivals and retrials occur simultaneously in $(t, t+)$. If a primary arrival and one or more retrials occur at the same time and there are not enough free servers, the primary arrival has priority over the repeated arrivals. We also assume random order discipline among the customers in the orbit to occupy the free servers.

The state of the system at time $t+$ may be described by the process $X_t = (C_t, N_t)$, where C_t corresponds to the number of busy servers at time $t+$ and N_t to the number of customers in orbit at that time. The process $\{X_t; t \geq 0\}$ is an irreducible Markov chain with state space $S = \{0, \dots, c\} \times \mathbb{Z}_+$. Artalejo et al. [18] proved that the Markov chain $\{X_t; t \geq 0\}$ is positive recurrent if and only if $p < cq$ and, consequently, to assure the existence of the steady-state we assume from now on that such a condition holds. The one-step transition probabilities, $P_{(i,j)(m,n)}$, are given by the following expressions [9,18]:

$$P_{(c,j)(c,j+1)} = p\bar{q}^c, \quad j \geq 0,$$

$$\begin{aligned} P_{(i,j)(m,n)} = {} & \left((1-\delta_{m,i+j-n+1})\bar{p} \binom{i}{i+j-m-n} q^{i+j-m-n} \bar{q}^{m+n-j} \right. \\ & \left. + (1-\delta_{j-n,m}) p \binom{i}{i+j+1-m-n} q^{i+j+1-m-n} \bar{q}^{m+n-j-1} \right) \\ & \times \left(\binom{j}{j-n} s^{j-n} \bar{s}^{n} + \delta_{mc}(1-\delta_{n0}) \sum_{k=j-n+1}^{j} \binom{j}{k} s^k \bar{s}^{j-k} \right), \end{aligned}$$

$$\max\{0, j-c\} \leq n \leq j, \quad j-n \leq m \leq c, \quad \max\{0, m+n-j-1\} \leq i \leq c,$$

where δ_{ab} denotes Kronecker's function defined by

$$\delta_{ab} = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

Since the number of customers in orbit may increase at most by 1 and decrease at most by $\min\{c, j\}$ at each slot, the one-step transition probability matrix $\mathbf{P} = (P_{(i,j)(m,n)})$ has the following structure:

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots \\ \mathbf{A}_{10} & \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \mathbf{A}_{c0} & \mathbf{A}_{c1} & \mathbf{A}_{c2} & \cdots & \mathbf{A}_{c,c+1} & \mathbf{0}_{c+1} & \cdots \\ \mathbf{0}_{c+1} & \mathbf{A}_{c+1,1} & \mathbf{A}_{c+1,2} & \cdots & \mathbf{A}_{c+1,c+1} & \mathbf{A}_{c+1,c+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\mathbf{A}_{jn}$ are square matrices of dimension $c + 1$ containing the transition probabilities from the jth level, $l(j) = \{(0, j), \ldots, (c, j)\},\ j \geq 0$, to the nth level, $l(n) = \{(0, n), \ldots, (c, n)\},\ n \geq 0$. The elements of the matrices $\mathbf{A}_{jn}$ will be denoted by $a_{jn}^{(i,m)}$, $0 \leq i \leq c$, $0 \leq m \leq c$. We notice that $\mathbf{A}_{jn} = \mathbf{0}_{c+1}$, for $n < j - c$ and $n > j + 1$, being $\mathbf{0}_{c+1}$ a square matrix of dimension $c + 1$ with every element equals 0.

More concretely, the matrices $\mathbf{A}_{jj}$ are characterized because the number of customers in the orbit remains constant:

$$\mathbf{A}_{jj} = \left(\mathbf{a}_{jj}^{(0)}, \ldots, \mathbf{a}_{jj}^{(c-1)}, \mathbf{a}_{jj}^{(c)}\right) = \left(\bar{s}^j \mathbf{a}_{00}^{(0)}, \ldots, \bar{s}^j \mathbf{a}_{00}^{(c-1)}, \mathbf{a}_{00}^{(c)}\right), \quad j \geq 0,$$

where

$$\mathbf{a}_{00}^{(m)} = (1 - \delta_{m0}) p \bar{q}^{m-1} \mathbf{v}_{m-1} + \bar{p} \bar{q}^m \mathbf{v}_m, \quad 0 \leq m \leq c,$$

with $\mathbf{v}_m$ a column vector of dimension $c + 1$ whose ith component is given by $\binom{i-1}{i-m-1} q^{i-m-1}$ if $m + 1 \leq i \leq c + 1$ and 0 otherwise. Hence the ith entry of the vector $\mathbf{v}_m$ when multiplied by $\bar{q}^m$ represents the probability of m servers remain busy while the remaining $i - m - 1$ servers become free, $m + 1 \leq i \leq c + 1$.

The matrices $\mathbf{A}_{j,j+1}$ are characterized because the number of customers in the orbit increases one unit:

$$\begin{aligned} \mathbf{A}_{j,j+1} &= \mathbf{A} = \left(\mathbf{a}_{j,j+1}^{(0)}, \ldots, \mathbf{a}_{j,j+1}^{(c-1)}, \mathbf{a}_{j,j+1}^{(c)}\right) \\ &= \left(\mathbf{0}(c+1), \ldots, \mathbf{0}(c+1), p\bar{q}^c \mathbf{e}_{c+1}(c+1)\right), \quad j \geq 0, \end{aligned}$$

being $\mathbf{0}_{(c+1)}$ a column vector of dimension $c+1$ with every element equal to 0 and $\mathbf{e}_j(k)$ a column vector of dimension k with 1 in the jth position and 0 otherwise. Thus $\mathbf{A}_{j,j+1}$ is a square matrix of dimension $c + 1$ with $p\bar{q}^c$ in the position $(c + 1, c + 1)$ and 0 otherwise, where $p\bar{q}^c$ indicates that all the servers remain busy while a customer arrives and consequently goes to the orbit.

The matrices $\mathbf{A}_{jn}$ are characterized because the number of customers in the orbit diminishes from j to n:

$$\begin{aligned} \mathbf{A}_{jn} &= \left(\mathbf{a}_{jn}^{(0)}, \ldots, \mathbf{a}_{jn}^{(c)}\right) = \binom{j}{j-n} s^{j-n} \bar{s}^n \left(\mathbf{0}(c+1), \ldots, \mathbf{0}(c+1), \mathbf{a}_{00}^{(0)}, \ldots, \mathbf{a}_{00}^{(c-j+n)}\right) \\ &\quad + (1 - \delta_{n0}) \sum_{k=j-n+1}^{j} \binom{j}{k} s^k \bar{s}^{j-k} \left(\mathbf{0}(c+1), \ldots, \mathbf{0}(c+1), \mathbf{a}_{00}^{(c-j+n)}\right), \quad j \geq 1,\ \max\{0, j - c\} \leq n \leq j - 1, \end{aligned}$$

where in the first term of the sum $j - n$ successful retrials occur and in the second one there are more retrials than idle servers.

We now define the performance descriptors of the customer's behavior: R^s, R^b, A^s and A^b are, respectively, the number of successful retrials, the number of blocked retrials, the number of successful primary arrivals and the number of blocked primary arrivals during a busy period. These four measures are referred to a busy period, which is defined as the first passage time to the state $(0, 0)$, given that the initial state is $(1, 0)$. Since $R^s = A^b$ [15], we will only have to study one of them, for example R^s.

Throughout the paper, we will denote $\mathbf{I}_k$ as the identity matrix of dimension k, $\mathbf{e}(k)$ as a column vector of dimension k with all the elements equal 1, and $\mathbf{E}_{ij}(k) = \mathbf{e}_i(k)\mathbf{e}_j(k)'$.

3. Probability mass function of the number of successful retrials

This section provides a recursive procedure to compute in an exact way the p.m.f. $P\{R^s = r\}$, $r \geq 0$. Let $x_{ij}^{R^s}(r)$ be the probability that $r \geq 0$ successful retrials take place during the remaining busy period, given that the current system state is (i, j), with $(i, j) \in S$. We observe that $x_{ij}^{R^s}(r) = 0$, $(i, j) \in S,\ j \geq r + 1$, and the probability distribution of R^s is given by $P\{R^s = r\} = x_{10}^{R^s}(r),\ r \geq 0$.

As usual, we group the probabilities $\{x_{ij}^{R^s}(r), (i,j) \in S,\ r \geq j\}$ according to the orbit level as follows

$$\mathbf{x}^{R^s}(r) = \left(\mathbf{x}_0^{R^s}(r), \ldots, \mathbf{x}_r^{R^s}(r)\right)', \quad r \geq 0, \qquad \mathbf{x}_j^{R^s}(r) = \left(x_{0j}^{R^s}(r), \ldots, x_{cj}^{R^s}(r)\right)', \quad 0 \leq j \leq r.$$

The next theorem gives an algorithmic solution for the probabilities $x_{ij}^{R^s}(r), (i,j) \in S,\ r \geq j$.

Theorem 1. *For $r = 0$, the probabilities $\mathbf{x}^{R^s}(0)$ can be computed as the solution of the system*

$$\mathbf{B}_{00}\mathbf{x}^{R^s}(0) = \mathbf{b}^{R^s}(0),$$

where

$$\mathbf{B}_{00} = \mathbf{A}_{00} - \mathbf{I}_{c+1} + \mathbf{E}_{11}(c+1) - p\mathbf{E}_{12}(c+1),$$
$$\mathbf{b}^{R^s}(0) = \bar{p}\mathbf{e}_1(c+1).$$

For each fixed $r \geq 1$, the probabilities $\mathbf{x}^{R^s}(r)$ can be computed recursively as the solution of the system

$$\mathbf{P}_{R^s}\mathbf{x}^{R^s}(r) = \mathbf{b}^{R^s}(r),$$

where

$$\mathbf{P}_{R^s} = \begin{pmatrix} \mathbf{B}_{00} & \mathbf{A} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} \\ \mathbf{0}_{c+1} & \mathbf{A}_{11} - \mathbf{I}_{c+1} & \mathbf{A} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{A}_{r-1,r-1} - \mathbf{I}_{c+1} & \mathbf{A} \\ \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} & \mathbf{A}_{rr} - \mathbf{I}_{c+1} \end{pmatrix},$$

$$\mathbf{b}^{R^s}(r) = \left(\mathbf{0}(c+1), \mathbf{b}_1^{R^s}(r), \ldots, \mathbf{b}_r^{R^s}(r)\right)',$$

$$\mathbf{b}_k^{R^s}(r) = -\sum_{j=1}^{\min\{c,k\}} \mathbf{A}_{k,k-j}\mathbf{x}_{k-j}^{R^s}(r-j), \quad 1 \leq k \leq r.$$

Proof. The proof is based on writing in matrix form the following system of equations for each $r \geq 0$:

$$x_{00}^{R^s}(r) = \delta_{r0},$$

$$x_{ij}^{R^s}(r) = \sum_{n=\max\{0,j-c\}}^{j} \sum_{m=0}^{c} a_{jn}^{(i,m)} x_{mn}^{R^s}(r-j+n) + (1-\delta_{rj})\delta_{ic}p\bar{q}^c x_{c,j+1}^{R^s}(r), \quad 0 \leq i \leq c,\ 0 \leq j \leq r,\ (i,j) \neq (0,0).$$

These equations have been obtained by conditioning on the next transition of the Markov chain (first-step analysis). □

It should be pointed out that the above theorem shows a recursive scheme to compute the exact probability distribution of R^s.

4. Probability mass function of the number of blocked retrials

In this section, we analyze the p.m.f. of R^b: $P\{R^b = r\}$, $r \geq 0$. The nature of the problem makes it necessary to approximate the *Geo*/*Geo*/*c* retrial queue with infinite orbit by the truncated model with finite orbit of capacity K, so the primary arrivals who find the system in the state (c, K) leave the system forever without receiving service. For convenience, we assume $K > c$. For this truncated model, the state space is $S^K = \{0, \ldots, c\} \times \{0, \ldots, K\}$ and the one-step transition probability matrix is given by

$$\mathbf{P}_K = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} \\ \mathbf{A}_{10} & \mathbf{A}_{11} & \mathbf{A} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{c0} & \mathbf{A}_{c1} & \mathbf{A}_{c2} & \cdots & \mathbf{A} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} \\ \mathbf{0}_{c+1} & \mathbf{A}_{c+1,1} & \mathbf{A}_{c+1,2} & \cdots & \mathbf{A}_{c+1,c+1} & \mathbf{A} & \cdots & \mathbf{0}_{c+1} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \mathbf{A}_{K-1,K-1-c} & & \mathbf{A}_{K-1,K-c} & & \cdots & \mathbf{A}_{K-1,K-1} & \mathbf{A} \\ \cdots & \mathbf{0}_{c+1} & & \mathbf{A}_{K,K-c} & & \cdots & \mathbf{A}_{K,K-1} & \mathbf{A}_{KK}^{*} \end{pmatrix},$$

where $\mathbf{A}_{KK}^{*} = \mathbf{A}_{KK} + \mathbf{A}$.

This method consisting in placing a fictitious limit K on the orbit capacity is known as the method of direct truncation. Such a method has been proved effective for dealing with both continuous-time retrial queues [9,10,15–17,19] and discrete-time retrial queues [9,18].

Let $x_{ij}^{R^b}(r)$ be the probability that $r \geq 0$ blocked retrials take place during the remaining busy period, given that the current system state is (i, j), with $(i, j) \in S^K$. We observe that the probability distribution of R^b is given by $P\{R^b = r\} = x_{10}^{R^b}(r), r \geq 0$.

As usual, we group the probabilities $\{x_{ij}^{R^b}(r), (i, j) \in S^K, r \geq 0\}$ with respect to the orbit level as follows

$$\mathbf{x}^{R^b}(r) = \left(\mathbf{x}_0^{R^b}(r), \dots, \mathbf{x}_K^{R^b}(r)\right)', \quad r \geq 0, \qquad \mathbf{x}_j^{R^b}(r) = \left(x_{0j}^{R^b}(r), \dots, x_{cj}^{R^b}(r)\right)', \quad 0 \leq j \leq K.$$

The following theorem provides an algorithm to calculate the probabilities $x_{ij}^{R^b}(r)$, $(i, j) \in S^K$, $r \geq 0$.

Theorem 2. *For each fixed $r \geq 0$, the probabilities $\mathbf{x}^{R^b}(r)$ can be computed recursively as the solution of the system*

$$(\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} - \mathbf{B}^{(0)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{x}^{R^b}(r) = \mathbf{b}^{R^b}(r),$$

where the matrix $\mathbf{B}^{(l)} = \left(\mathbf{B}_{jn}^{(l)}\right)$, $0 \leq j \leq K$, $0 \leq n \leq K$, has a block tridiagonal matrix structure with

$$\mathbf{B}_{jn}^{(0)} = \begin{cases} \displaystyle\sum_{l=j-n+1}^{j} \binom{j}{l} s^l \bar{s}^{j-l} \left(\mathbf{0}(c+1), \dots, \mathbf{0}(c+1), \mathbf{a}_{00}^{(c-j+n)}\right), \\ \quad 1 \leq j \leq K, \ \max\{1, j-c\} \leq n \leq j, \ (j, n) \neq (K, K), \\ p\bar{q}^c (1 - \bar{s}^j) \left(\mathbf{0}(c+1), \dots, \mathbf{0}(c+1), \mathbf{e}_{c+1}(c+1)\right), \quad 0 \leq j \leq K-1, \ n = j+1, \\ (1 - \bar{s}^K) \left(\mathbf{0}(c+1), \dots, \mathbf{0}(c+1), \mathbf{a}_{00}^{(c)} + p\bar{q}^c \mathbf{e}_{c+1}(c+1)\right), \quad (j, n) = (K, K), \\ \mathbf{0}_{c+1}, \quad \textit{otherwise}, \end{cases}$$

for $1 \leq l \leq K-1$,

$$\mathbf{B}_{jn}^{(l)} = \begin{cases} \displaystyle\binom{j}{l+j-n} s^{l+j-n} \bar{s}^{n-l} \left(\mathbf{0}(c+1), \dots, \mathbf{0}(c+1), \mathbf{a}_{00}^{(c-j+n)}\right), \\ \quad l \leq j \leq K, \ \max\{l, j-c\} \leq n \leq j, \ (j, n) \neq (K, K), \\ p\bar{q}^c \displaystyle\binom{j}{l} s^l \bar{s}^{j-l} \left(\mathbf{0}(c+1), \dots, \mathbf{0}(c+1), \mathbf{e}_{c+1}(c+1)\right), \quad l \leq j \leq K-1, \ n = j+1, \\ \displaystyle\binom{K}{l} s^l \bar{s}^{K-l} \left(\mathbf{0}(c+1), \dots, \mathbf{0}(c+1), \mathbf{a}_{00}^{(c)} + p\bar{q}^c \mathbf{e}_{c+1}(c+1)\right), \quad (j, n) = (K, K), \\ \mathbf{0}_{c+1}, \quad \textit{otherwise}, \end{cases}$$

$$\mathbf{B}_{jn}^{(K)} = \begin{cases} s^K \left(\mathbf{0}(c+1), \dots, \mathbf{0}(c+1), \mathbf{a}_{00}^{(c)} + p\bar{q}^c \mathbf{e}_{c+1}(c+1)\right), & (j, n) = (K, K), \\ \mathbf{0}_{c+1}, & \textit{otherwise}, \end{cases}$$

$$\mathbf{b}^{R^b}(r) = \begin{cases} \bar{p}\mathbf{e}_1((K+1)(c+1)), & r = 0, \\ -\displaystyle\sum_{l=1}^{\min\{K,r\}} \mathbf{B}^{(l)} \mathbf{x}^{R^b}(r-l), & r \geq 1. \end{cases}$$

Proof. For each fixed $r \geq 0$, taking into account a first-step analysis, we have:

$$x_{00}^{R^b}(r) = \delta_{r0}, \tag{1}$$

$$\begin{aligned} x_{ij}^{R^b}(r) &= \sum_{n=\max\{0,j-c\}}^{j} \sum_{m=0}^{c-1} a_{jn}^{(i,m)} x_{mn}^{R^b}(r) + \sum_{n=\max\{0,j-c\}}^{j} a_{00}^{(i,c-j+n)} \sum_{l=0}^{\min\{n,r\}} \binom{j}{l+j-n} s^{l+j-n} \bar{s}^{n-l} x_{cn}^{R^b}(r-l) \\ &\quad + \delta_{ic} p\bar{q}^c \sum_{l=0}^{\min\{j,r\}} \binom{j}{l} s^l \bar{s}^{j-l} x_{c,j+1}^{R^b}(r-l), \quad 0 \leq i \leq c, \ 0 \leq j \leq K, \ (i, j) \notin \{(0, 0), (c, K)\}, \end{aligned} \tag{2}$$

$$\begin{aligned} x_{cK}^{R^b}(r) &= \sum_{n=K-c}^{K} \sum_{m=0}^{c-1} a_{Kn}^{(c,m)} x_{mn}^{R^b}(r) + \sum_{n=K-c}^{K} a_{00}^{(c,c-K+n)} \sum_{l=0}^{\min\{n,r\}} \binom{K}{l+K-n} s^{l+K-n} \bar{s}^{n-l} x_{cn}^{R^b}(r-l) \\ &\quad + p\bar{q}^c \sum_{l=0}^{\min\{K,r\}} \binom{K}{l} s^l \bar{s}^{K-l} x_{cK}^{R^b}(r-l). \end{aligned}$$

The proof finishes setting in matrix form the preceding system of equations. $\square$

To complete this section, we discuss the need to work with the truncated model. For the original model with infinite orbit, the probabilities $\{x_{ij}^{R^b}(r), (i,j) \in S, r \geq 0\}$ satisfy the Eqs. (1) for $(i,j) = (0,0)$ and (2) for $(i,j) \neq (0,0)$. Under these conditions, for each fixed $r \geq 0$, we have an infinite system of equations, which has no known solution. That is why, we find the approximate solution for the p.m.f. of R^b by means of truncating the orbit capacity.

5. Probability mass function of the number of successful arrivals

This section pays attention to the study of the p.m.f. $P\{A^s = a\}, a \geq 0$. In this case, the difficulty of finding the exact distribution of A^s also becomes evident. To avoid this problem, we again propose to consider the truncated model with orbit of finite capacity $K > c$.

Let $x_{ij}^{A^s}(a)$ be the probability that $a \geq 0$ successful primary arrivals occur during the remaining busy period, given that the current system state is (i,j), with $(i,j) \in S^K$. We notice that the probability distribution of A^s is given by $P\{A^s = a\} = x_{10}^{A^s}(a), a \geq 0$.

As previously, we group the probabilities $\{x_{ij}^{A^s}(a), (i,j) \in S^K, a \geq 0\}$ in accordance with the orbit level as follows

$$\mathbf{x}^{A^s}(a) = \left(\mathbf{x}_0^{A^s}(a), \ldots, \mathbf{x}_K^{A^s}(a)\right)', \quad a \geq 0, \qquad \mathbf{x}_j^{A^s}(a) = \left(x_{0j}^{A^s}(a), \ldots, x_{cj}^{A^s}(a)\right)', \quad 0 \leq j \leq K.$$

Theorem 3 presents an algorithm to calculate the probabilities $x_{ij}^{A^s}(a), (i,j) \in S^K, a \geq 0$.

Theorem 3. *For each fixed $a \geq 0$, the probabilities $\mathbf{x}^{A^s}(a)$ can be computed recursively as the solution of the system*

$$(\mathbf{P}_K - \mathbf{G}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)))\mathbf{x}^{A^s}(a) = \mathbf{b}^{A^s}(a),$$

where

$$\mathbf{G}_K = \begin{pmatrix} \mathbf{G}_{00} & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} \\ \mathbf{G}_{10} & \mathbf{G}_{11} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{c0} & \mathbf{G}_{c1} & \mathbf{G}_{c2} & \cdots & \mathbf{0}_{c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} \\ \mathbf{0}_{c+1} & \mathbf{G}_{c+1,1} & \mathbf{G}_{c+1,2} & \cdots & \mathbf{G}_{c+1,c+1} & \mathbf{0}_{c+1} & \cdots & \mathbf{0}_{c+1} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \mathbf{G}_{K-1,K-1-c} & & \mathbf{G}_{K-1,K-c} & & \cdots & \mathbf{G}_{K-1,K-1} & \mathbf{0}_{c+1} \\ \cdots & \mathbf{0}_{c+1} & & \mathbf{G}_{K,K-c} & & \cdots & \mathbf{G}_{K,K-1} & \mathbf{G}_{KK} \end{pmatrix},$$

$$\mathbf{G}_{jn} = \left(\mathbf{g}_{jn}^{(0)}, \ldots, \mathbf{g}_{jn}^{(c)}\right), \qquad \mathbf{g}_{jn}^{(m)} = \left(g_{jn}^{(0,m)}, \ldots, g_{jn}^{(c,m)}\right)', \quad 0 \leq j \leq K,\ \max\{0, j-c\} \leq n \leq j,\ 0 \leq m \leq c,$$

$$\mathbf{g}_{00}^{(m)} = (1 - \delta_{m0})p\bar{q}^{m-1}\mathbf{v}_{m-1}, \quad 0 \leq m \leq c,$$

$$\mathbf{G}_{jj} = \left(\bar{s}^j \mathbf{g}_{00}^{(0)}, \ldots, \bar{s}^j \mathbf{g}_{00}^{(c-1)}, \mathbf{g}_{00}^{(c)}\right), \quad 0 \leq j \leq K,$$

$$\begin{aligned} \mathbf{G}_{jn} = {} & \binom{j}{j-n} s^{j-n}\bar{s}^n \left(\mathbf{0}(c+1), \ldots, \mathbf{0}(c+1), \mathbf{g}_{00}^{(0)}, \ldots, \mathbf{g}_{00}^{(c-j+n)}\right) \\ & + (1 - \delta_{n0}) \sum_{k=j-n+1}^{j} \binom{j}{k} s^k \bar{s}^{j-k} \left(\mathbf{0}(c+1), \ldots, \mathbf{0}(c+1), \mathbf{g}_{00}^{(c-j+n)}\right), \quad 1 \leq j \leq K,\ \max\{0, j-c\} \leq n \leq j-1, \end{aligned}$$

$$\mathbf{b}^{A^s}(a) = \begin{cases} \bar{p}\mathbf{e}_1((K+1)(c+1)), & a = 0, \\ (-\mathbf{G}_K + p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{x}^{A^s}(a-1), & a \geq 1. \end{cases}$$

Proof. The proof begins by writing the matrices $\mathbf{A}_{jn}$, $0 \leq j \leq K$, $\max\{0, j-c\} \leq n \leq j$, as the sum of two squares matrices $\mathbf{G}_{jn}$ and $\mathbf{H}_{jn}$. The matrices $\mathbf{G}_{jn}$ are associated to the successful primary arrivals and their definition appears in the statement of the theorem, while the matrices $\mathbf{H}_{jn}$ are defined as follows

$$\mathbf{H}_{jn} = \left(\mathbf{h}_{jn}^{(0)}, \ldots, \mathbf{h}_{jn}^{(c)}\right), \qquad \mathbf{h}_{jn}^{(m)} = \left(h_{jn}^{(0,m)}, \ldots, h_{jn}^{(c,m)}\right)', \quad 0 \leq m \leq c,$$

$$\mathbf{h}_{00}^{(m)} = \mathbf{a}_{00}^{(m)} - \mathbf{g}_{00}^{(m)} = \bar{p}\bar{q}^m \mathbf{v}_m, \quad 0 \leq m \leq c,$$

$$\mathbf{H}_{jj} = \mathbf{A}_{jj} - \mathbf{G}_{jj}, \quad 0 \leq j \leq K, \qquad \mathbf{H}_{KK} = \mathbf{A}_{KK}^* - \mathbf{A} - \mathbf{G}_{KK},$$

$$\mathbf{H}_{jn} = \mathbf{A}_{jn} - \mathbf{G}_{jn}, \quad 1 \leq j \leq K,\ \max\{0, j-c\} \leq n \leq j-1.$$

For each fixed $a \geq 0$, we now carry out a first-step analysis to obtain:

$$x_{00}^{A^s}(a) = \delta_{a0}, \tag{3}$$

$$\begin{aligned} x_{ij}^{A^s}(a) \;=\; & (1-\delta_{a0}) \sum_{n=\max\{0,j-c\}}^{j} \sum_{m=0}^{c} g_{jn}^{(i,m)} x_{mn}^{A^s}(a-1) + \sum_{n=\max\{0,j-c\}}^{j} \sum_{m=0}^{c} h_{jn}^{(i,m)} x_{mn}^{A^s}(a) \\ & + \delta_{ic} p\bar{q}^c x_{c,j+1}^{A^s}(a), \quad 0 \leq i \leq c,\ 0 \leq j \leq K,\ (i,j) \notin \{(0,0),(c,K)\}, \end{aligned} \tag{4}$$

$$x_{cK}^{A^s}(a) = (1-\delta_{a0}) \sum_{n=K-c}^{K} \sum_{m=0}^{c} g_{Kn}^{(c,m)} x_{mn}^{A^s}(a-1) + \sum_{n=K-c}^{K} \sum_{m=0}^{c} h_{Kn}^{(c,m)} x_{mn}^{A^s}(a) + p\bar{q}^c x_{cK}^{A^s}(a).$$

The proof concludes after putting in matrix form the preceding system of equations. $\square$

We conclude this section with a small talk about the necessity of considering the truncated model in order to compute the p.m.f. of A^s. As in the previous section, the formulas (3) and (4) shed light in that sense.

6. Joint generating functions and moments

In this section we are interested in calculating the moments of the descriptors under study, such as the covariance structure among them. More specifically, we want to compute the expectation and the coefficient of variation of each descriptor, $E[R^s]$, $E[R^b]$, $E[A^s]$, $C_v(R^s)$, $C_v(R^b)$, $C_v(A^s)$, and the coefficients of correlation $\rho(R^s, R^b)$, $\rho(R^s, A^s)$, $\rho(R^b, A^s)$, where

$$C_v(R^s) = \frac{\sigma(R^s)}{E[R^s]},$$

$$\rho(R^s, R^b) = \frac{E[R^s R^b] - E[R^s]E[R^b]}{\sigma(R^s)\sigma(R^b)},$$

and $\sigma(R^s)$, $\sigma(R^b)$, $\sigma(A^s)$ denote the standard deviations of the measures under consideration.

The problem reduces to the calculation of the first moments, the second moments and the cross moments. In order to obtain these characteristics, we consider the truncated model with finite orbit of capacity $K > c$. For $0 \leq i \leq c$ and $0 \leq j \leq K$, let $\Phi_{ij}(z_1, z_2, z_3)$ be the joint generating function defined by

$$\Phi_{ij}(z_1, z_2, z_3) = E\left[z_1^{R_{ij}^s} z_2^{R_{ij}^b} z_3^{A_{ij}^s} \right], \quad |z_k| \leq 1,\ 1 \leq k \leq 3,$$

where R_{ij}^s, R_{ij}^b and A_{ij}^s are, respectively, the number of successful retrials, the number of blocked retrials and the number of successful primary arrivals during the residual busy period, given that the current state is $(i,j) \in S^K$. Obviously $R^s = R_{10}^s$ and the same for the other descriptors.

Using a first-step analysis, we get the following set of equations:

$$\Phi_{00}(z_1, z_2, z_3) = 1, \tag{5}$$

$$\begin{aligned} \Phi_{ij}(z_1, z_2, z_3) \;=\; & \sum_{n=\max\{0,j-c\}}^{j} \sum_{l=0}^{c-1} z_1^{j-n} \left(z_3 g_{jn}^{(i,l)} + h_{jn}^{(i,l)} \right) \Phi_{ln}(z_1, z_2, z_3) \\ & + \sum_{n=\max\{0,j-c\}}^{j} z_1^{j-n} \left(z_3 g_{00}^{(i,c-j+n)} + h_{00}^{(i,c-j+n)} \right) \sum_{k=0}^{n} z_2^k \binom{j}{k+j-n} s^{k+j-n} \bar{s}^{n-k} \Phi_{cn}(z_1, z_2, z_3) \\ & + \delta_{ic} p\bar{q}^c \sum_{k=0}^{j} z_2^k \binom{j}{k} s^k \bar{s}^{j-k} \Phi_{c,j+1}(z_1, z_2, z_3), \quad 0 \leq i \leq c,\ 0 \leq j \leq K,\ (i,j) \notin \{(0,0),(c,K)\}, \end{aligned} \tag{6}$$

$$\begin{aligned} \Phi_{cK}(z_1, z_2, z_3) \;=\; & \sum_{n=K-c}^{K} \sum_{l=0}^{c-1} z_1^{K-n} \left(z_3 g_{Kn}^{(c,l)} + h_{Kn}^{(c,l)} \right) \Phi_{ln}(z_1, z_2, z_3) \\ & + \sum_{n=K-c}^{K} z_1^{K-n} \left(z_3 g_{00}^{(c,c-K+n)} + h_{00}^{(c,c-K+n)} \right) \sum_{k=0}^{n} z_2^k \binom{K}{k+K-n} s^{k+K-n} \bar{s}^{n-k} \Phi_{cn}(z_1, z_2, z_3) \\ & + p\bar{q}^c \sum_{k=0}^{K} z_2^k \binom{K}{k} s^k \bar{s}^{K-k} \Phi_{cK}(z_1, z_2, z_3), \end{aligned} \tag{7}$$

where the probabilities $h_{jn}^{(i,m)}$, $0 \leq i \leq c$, $0 \leq j \leq K$, $0 \leq m \leq c$, $\max\{0, j-c\} \leq n \leq j$ are defined in the proof of Theorem 3.

To calculate the moments of the descriptors, we introduce the following notation:

$$m_{ij}^{R^s} = E\left[R_{ij}^s\right], \qquad m_{ij}^{R^b} = E\left[R_{ij}^b\right], \qquad m_{ij}^{A^s} = E\left[A_{ij}^s\right], \quad (i,j) \in S^K,$$
$$m_{ij}^{R^s,2} = E\left[(R_{ij}^s)^2\right], \qquad m_{ij}^{R^b,2} = E\left[(R_{ij}^b)^2\right], \qquad m_{ij}^{A^s,2} = E\left[(A_{ij}^s)^2\right], \quad (i,j) \in S^K,$$
$$m_{ij}^{R^s,R^b} = E\left[R_{ij}^s R_{ij}^b\right], \qquad m_{ij}^{R^s,A^s} = E\left[R_{ij}^s A_{ij}^s\right], \qquad m_{ij}^{R^b,A^s} = E\left[R_{ij}^b A_{ij}^s\right], \quad (i,j) \in S^K.$$

The following vectors comprise the moments partitioned according to the orbit levels:

$$\mathbf{m}^{R^s} = \left(\mathbf{m}_0^{R^s}, \ldots, \mathbf{m}_K^{R^s}\right)', \qquad \mathbf{m}_j^{R^s} = \left(m_{0j}^{R^s}, \ldots, m_{cj}^{R^s}\right)', \quad 0 \le j \le K,$$
$$\mathbf{m}^{R^s,2} = \left(\mathbf{m}_0^{R^s,2}, \ldots, \mathbf{m}_K^{R^s,2}\right)', \qquad \mathbf{m}_j^{R^s,2} = \left(m_{0j}^{R^s,2}, \ldots, m_{cj}^{R^s,2}\right)', \quad 0 \le j \le K,$$
$$\mathbf{m}^{R^s,R^b} = \left(\mathbf{m}_0^{R^s,R^b}, \ldots, \mathbf{m}_K^{R^s,R^b}\right)', \qquad \mathbf{m}_j^{R^s,R^b} = \left(m_{0j}^{R^s,R^b}, \ldots, m_{cj}^{R^s,R^b}\right)', \quad 0 \le j \le K.$$

By analogy the vectors comprising the moments for the other descriptors have a similar notation.

Theorems 4–6 give the systems of equations for computing the moments of the number of successful retrials, blocked retrials and successful primary arrivals.

Theorem 4. *The first moments* $\{m_{ij}^{R^s},\ m_{ij}^{R^b},\ m_{ij}^{A^s},\ 0 \le i \le c,\ 0 \le j \le K\}$ *verify the systems of equations*

$$(\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^s} = \mathbf{d}^{R^s}, \tag{8}$$
$$(\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^b} = \mathbf{d}^{R^b},$$
$$(\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{A^s} = \mathbf{d}^{A^s},$$

where

$$\mathbf{d}^{R^s} = -\tilde{\mathbf{A}}\mathbf{e}((K+1)(c+1)),$$
$$\mathbf{d}^{R^b} = -\sum_{l=1}^{K} \mathbf{B}^{(l)} l\mathbf{e}((K+1)(c+1)),$$
$$\mathbf{d}^{A^s} = (-\mathbf{G}_K + p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{e}((K+1)(c+1)),$$

and the matrix $\tilde{\mathbf{A}} = (\tilde{\mathbf{A}}_{jn})$, $0 \le j \le K$, $0 \le n \le K$, *has a block tridiagonal matrix structure with*

$$\tilde{\mathbf{A}}_{jn} = \begin{cases} (j-n)\mathbf{A}_{jn}, & 1 \le j \le K, \max\{0, j-c\} \le n \le j-1, \\ \mathbf{0}_{c+1}, & \text{otherwise}. \end{cases}$$

Proof. We present the proof for R^s and omit the proofs for R^b and A^s, because they follow similar arguments to those given for R^s.

Since $m_{ij}^{R^s} = \frac{\partial}{\partial z_1}\Phi_{ij}(z_1,z_2,z_3)\Big|_{(1,1,1)}$, $0 \le i \le c$, $0 \le j \le K$, if we differentiate Eqs. (5)–(7) with respect to z_1 and set $(z_1,z_2,z_3) = (1,1,1)$, we get the equations governing the dynamic of the moments $\{m_{ij}^{R^s}, (i,j) \in S^K\}$:

$$m_{00}^{R^s} = 0, \tag{9}$$
$$m_{ij}^{R^s} = \sum_{n=\max\{0,j-c\}}^{j} \sum_{l=0}^{c} a_{jn}^{(i,l)}(m_{ln}^{R^s} + j - n) + \delta_{ic} p\bar{q}^c m_{c,j+1}^{R^s}, \quad 0 \le i \le c,\ 0 \le j \le K,\ (i,j) \notin \{(0,0),(c,K)\}, \tag{10}$$
$$m_{cK}^{R^s} = \sum_{n=K-c}^{K} \sum_{l=0}^{c} a_{Kn}^{(c,l)}(m_{ln}^{R^s} + K - n) + p\bar{q}^c m_{cK}^{R^s}. \tag{11}$$

It can be verified that the Eqs. (9)–(11) can be expressed in matrix form as (8). $\square$

Theorem 5. *The second moments* $\{m_{ij}^{R^s,2},\ m_{ij}^{R^b,2},\ m_{ij}^{A^s,2},\ 0 \le i \le c,\ 0 \le j \le K\}$ *verify the systems of equations*

$$(\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^s,2} = \mathbf{f}^{R^s}, \tag{12}$$
$$(\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^b,2} = \mathbf{f}^{R^b},$$
$$(\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{A^s,2} = \mathbf{f}^{A^s},$$

where

$$\mathbf{f}^{R^s} = -2\tilde{\mathbf{A}}\mathbf{m}^{R^s} - \hat{\mathbf{A}}\mathbf{e}((K+1)(c+1)),$$
$$\mathbf{f}^{R^b} = -\sum_{l=1}^{K}\mathbf{B}^{(l)}(2l\mathbf{m}^{R^b} + l^2\mathbf{e}((K+1)(c+1))),$$
$$\mathbf{f}^{A^s} = (-\mathbf{G}_K + p\mathbf{E}_{12}((K+1)(c+1)))(2\mathbf{m}^{A^s} + \mathbf{e}((K+1)(c+1))),$$

and the matrix $\hat{\mathbf{A}} = (\hat{\mathbf{A}}_{jn}),\ 0 \le j \le K, 0 \le n \le K$, *has a block tridiagonal matrix structure with*

$$\hat{\mathbf{A}}_{jn} = \begin{cases}(j-n)^2\mathbf{A}_{jn}, & 1 \le j \le K, \max\{0, j-c\} \le n \le j-1,\\ \mathbf{0}_{c+1}, & \textit{otherwise}.\end{cases}$$

Proof. As in the proof of Theorem 4, we present the proof for R^s and omit the proofs for R^b and A^s, because they follow similar arguments to those given for R^s.

We use $\frac{\partial^2}{\partial z_1^2}\Phi_{ij}(z_1, z_2, z_3)\Big|_{(1,1,1)} = m_{ij}^{R^s,2} - m_{ij}^{R^s}$ to obtain the equations for $\{m_{ij}^{R^s,2}, (i,j) \in S^K\}$. By differentiating the Eqs. (5)–(7) twice with respect to z_1, setting $(z_1, z_2, z_3) = (1, 1, 1)$ and using (10) and (11), we get

$$m_{00}^{R^s,2} = 0, \tag{13}$$

$$\begin{aligned} m_{ij}^{R^s,2} = & \sum_{n=\max\{0,j-c\}}^{j}\sum_{l=0}^{c} a_{jn}^{(i,l)}(m_{ln}^{R^s,2} + 2(j-n)m_{ln}^{R^s} + (j-n)^2) \\ & + \delta_{ic}p\bar{q}^c m_{c,j+1}^{R^s,2}, \quad 0 \le i \le c,\ 0 \le j \le K,\ (i,j) \notin \{(0,0),(c,K)\}, \end{aligned} \tag{14}$$

$$m_{cK}^{R^s,2} = \sum_{n=K-c}^{K}\sum_{l=0}^{c} a_{Kn}^{(c,l)}(m_{ln}^{R^s,2} + 2(K-n)m_{ln}^{R^s} + (K-n)^2) + p\bar{q}^c m_{cK}^{R^s,2}. \tag{15}$$

Putting Eqs. (13)–(15) in matrix form yields (12). □

Theorem 6. *The cross moments* $\{m_{ij}^{R^s,R^b}, m_{ij}^{R^s,A^s}, m_{ij}^{R^b,A^s},\ 0 \le i \le c,\ 0 \le j \le K\}$ *verify the systems of equations*

$$\begin{aligned} (\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^s,R^b} &= \mathbf{d}^{R^s,R^b}, \\ (\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^s,A^s} &= \mathbf{d}^{R^s,A^s}, \\ (\mathbf{P}_K - \mathbf{I}_{(K+1)(c+1)} + \mathbf{E}_{11}((K+1)(c+1)) - p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^b,A^s} &= \mathbf{d}^{R^b,A^s}, \end{aligned} \tag{16}$$

where

$$\mathbf{d}^{R^s,R^b} = -\sum_{l=1}^{K} l\mathbf{B}^{(l)}\mathbf{m}^{R^s} - \tilde{\mathbf{A}}\mathbf{m}^{R^b} - \sum_{l=1}^{K} l\tilde{\mathbf{B}}^{(l)}\mathbf{e}((K+1)(c+1)),$$
$$\mathbf{d}^{R^s,A^s} = (-\mathbf{G}_K + p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^s} - \tilde{\mathbf{A}}\mathbf{m}^{A^s} - \tilde{\mathbf{G}}\mathbf{e}((K+1)(c+1)),$$
$$\mathbf{d}^{R^b,A^s} = (-\mathbf{G}_K + p\mathbf{E}_{12}((K+1)(c+1)))\mathbf{m}^{R^b} - \sum_{l=1}^{K} l\mathbf{B}^{(l)}\mathbf{m}^{A^s} - \sum_{l=1}^{K} l\mathbf{G}^{(l)}\mathbf{e}((K+1)(c+1)),$$

and the matrices $\tilde{\mathbf{G}} = (\tilde{\mathbf{G}}_{jn}), \tilde{\mathbf{B}}^{(l)} = \left(\tilde{\mathbf{B}}_{jn}^{(l)}\right), \mathbf{G}^{(l)} = \left(\mathbf{G}_{jn}^{(l)}\right),\ 0 \le j \le K,\ 0 \le n \le K$, *have a block tridiagonal matrix structure with*

$$\tilde{\mathbf{G}}_{jn} = \begin{cases}(j-n)\mathbf{G}_{jn}, & 1 \le j \le K, \max\{0, j-c\} \le n \le j-1,\\ \mathbf{0}_{c+1}, & \textit{otherwise},\end{cases}$$

and for $1 \le l \le K$

$$\tilde{\mathbf{B}}_{jn}^{(l)} = \begin{cases}(j-n)\binom{j}{l+j-n}s^{l+j-n}\bar{s}^{n-l}\left(\mathbf{0}(c+1), \ldots, \mathbf{0}(c+1), \mathbf{a}_{00}^{(c-j+n)}\right), & l \le j \le K,\ \max\{l, j-c\} \le n \le j,\\ \mathbf{0}_{c+1}, & \textit{otherwise},\end{cases}$$
$$\mathbf{G}_{jn}^{(l)} = \begin{cases}\binom{j}{l+j-n}s^{l+j-n}\bar{s}^{n-l}\left(\mathbf{0}(c+1), \ldots, \mathbf{0}(c+1), \mathbf{g}_{00}^{(c-j+n)}\right), & l \le j \le K,\ \max\{l, j-c\} \le n \le j,\\ \mathbf{0}_{c+1}, & \textit{otherwise}.\end{cases}$$

Table 1
Main characteristics of the descriptor R^s.

		$s = 0.1$	$s = 0.3$	$s = 0.5$	$s = 0.7$	$s = 0.9$
$p = 0.2$	K	6	6	6	6	6
	$E[R^s]$	0.0014	0.0014	0.0014	0.0014	0.0015
	K	7	7	7	8	7
	$C_v(R^s)$	30.1195	30.4166	30.6021	30.7297	30.8228
$p = 0.4$	K	6	6	6	8	6
	$E[R^s]$	0.0992	0.1012	0.1034	0.1051	0.1063
	K	10	10	10	9	9
	$C_v(R^s)$	4.7909	4.8318	4.8634	4.8830	4.8956
$p = 0.6$	K	15	13	12	17	14
	$E[R^s]$	2.5887	2.3221	2.3218	2.3402	2.3590
	K	15	14	14	12	13
	$C_v(R^s)$	1.8833	1.8527	1.8492	1.8478	1.8468
$p = 0.8$	K	37	31	33	30	29
	$E[R^s]$	129.5344	66.1692	59.1919	56.9738	56.0580
	K	25	28	29	21	21
	$C_v(R^s)$	1.2057	1.1962	1.1928	1.1911	1.1902

Proof. We now turn our attention to the cross moments $\{m_{ij}^{R^s,R^b}, (i,j) \in S^K\}$. If we differentiate Eqs. (5)–(7) with respect to z_1 and z_2, and set $(z_1, z_2, z_3) = (1, 1, 1)$, then we get the following equations for the cross moments:

$$m_{00}^{R^s,R^b} = 0, \tag{17}$$

$$\begin{aligned} m_{ij}^{R^s,R^b} &= \sum_{n=\max\{0,j-c\}}^{j} \sum_{l=0}^{c} a_{jn}^{(i,l)} \left(m_{ln}^{R^s,R^b} + (j-n) m_{ln}^{R^b} \right) \\ &\quad + (1-\delta_{j0}) \sum_{n=\max\{1,j-c\}}^{j} a_{00}^{(i,c-j+n)} \sum_{l=1}^{n} l \binom{j}{l+j-n} s^{l+j-n} \bar{s}^{n-l} \left(m_{cn}^{R^s} + j - n \right) \\ &\quad + \delta_{ic} p \bar{q}^c \left(m_{c,j+1}^{R^s,R^b} + js m_{c,j+1}^{R^s} \right), \quad 0 \le i \le c,\ 0 \le j \le K,\ (i,j) \notin \{(0,0),(c,K)\}, \end{aligned} \tag{18}$$

$$\begin{aligned} m_{cK}^{R^s,R^b} &= \sum_{n=K-c}^{K} \sum_{l=0}^{c} a_{Kn}^{(c,l)} \left(m_{ln}^{R^s,R^b} + (K-n) m_{ln}^{R^b} \right) \\ &\quad + \sum_{n=K-c}^{K} a_{00}^{(c,c-K+n)} \sum_{l=1}^{n} l \binom{K}{l+K-n} s^{l+K-n} \bar{s}^{n-l} \left(m_{cn}^{R^s} + K - n \right) + p \bar{q}^c \left(m_{cK}^{R^s,R^b} + Ksm_{cK}^{R^s} \right). \end{aligned} \tag{19}$$

Expressing the system of Eqs. (17)–(19) in matrix form, the formula (16) is obtained. The proof corresponding to the cross moments $\{m_{ij}^{R^s,A^s}, (i,j) \in S^K\}$ and $\{m_{ij}^{R^b,A^s}, (i,j) \in S^K\}$ is similar and thus omitted. □

7. Numerical results

In this section, we present numerical results concerning the behavior of the descriptors under study. In addition, a simple optimization problem is solved.

To carry out this numerical analysis, the values are repeatedly computed as the solution of several systems of linear algebraic equations. The matrices of such systems are non-singular because their determinants are proportional to the determinants of irreducibly diagonally dominant matrices [24].

In order to obtain several moments of R^s, R^b and A^s, we need to determine the truncation level for the orbit capacity, K. For the calculations we use the following criterion in selecting the threshold K: we start with an initial value $K = c + 1$ and progressively increase the value of K until the first four decimal digits of the characteristic match.

First, we concentrate on the computation of the expectation and the coefficient of variation of R^s, R^b and A^s. Then, we calculate the correlation coefficients between R^s and R^b, between R^s and A^s and between R^b and A^s.

In all of these cases, we consider a queue with five servers ($c = 5$) and departure probability $q = 0.2$. The arrival probability p and the retrial probability s take values 0.2, 0.4, 0.6 and 0.8, and 0.1, 0.3, 0.5, 0.7 and 0.9, respectively.

Tables 1–3 show results about the expectations and the coefficients of variation of R^s, R^b and A^s, respectively, as well as their corresponding truncation levels, K. In the three tables we can observe that the truncation levels are increasing functions of p. This is a logical feature, because the more customers arrive at the system, the more congested it is, so a higher value of K is necessary to be accurately approximated. The expectations of R^s, R^b and A^s are increasing functions of p, for each fixed value of s. This agrees with our expectation, since the length of the busy period increases with p and, as a result, the number of events during a busy period also increases. The coefficients of variation of R^s, R^b and A^s are decreasing functions of p.

Table 2
Main characteristics of the descriptor R^b.

		$s = 0.1$	$s = 0.3$	$s = 0.5$	$s = 0.7$	$s = 0.9$
$p = 0.2$	K	6	6	6	6	6
	$E[R^b]$	0.0001	0.0003	0.0006	0.0008	0.0011
	K	9	9	10	9	9
	$C_v(R^b)$	101.3726	73.6622	67.3546	64.8268	63.6363
$p = 0.4$	K	6	6	7	7	7
	$E[R^b]$	0.0179	0.0464	0.0745	0.1027	0.1314
	K	14	13	13	14	13
	$C_v(R^b)$	11.0399	9.7198	9.5055	9.4736	9.4918
$p = 0.6$	K	15	15	15	15	16
	$E[R^b]$	1.0551	2.0108	3.0358	4.0958	5.1988
	K	19	18	20	20	21
	$C_v(R^b)$	2.8149	2.8725	2.9131	2.9461	2.9693
$p = 0.8$	K	42	38	39	39	40
	$E[R^b]$	144.8525	135.0813	177.2576	227.3547	282.0336
	K	33	32	32	33	33
	$C_v(R^b)$	1.3031	1.3871	1.4222	1.4414	1.4515

Table 3
Main characteristics of the descriptor A^s.

		$s = 0.1$	$s = 0.3$	$s = 0.5$	$s = 0.7$	$s = 0.9$
$p = 0.2$	K	6	6	6	6	6
	$E[A^s]$	1.8920	1.8905	1.8903	1.8902	1.8902
	K	6	6	6	6	6
	$C_v(A^s)$	1.5030	1.5010	1.5008	1.5007	1.5007
$p = 0.4$	K	6	6	6	6	7
	$E[A^s]$	8.9597	8.7598	8.7266	8.7138	8.7072
	K	6	6	6	6	6
	$C_v(A^s)$	1.3362	1.3140	1.3112	1.3103	1.3098
$p = 0.6$	K	15	13	12	15	12
	$E[A^s]$	49.5413	42.2156	41.0009	40.5309	40.2910
	K	9	7	8	6	7
	$C_v(A^s)$	1.2541	1.2024	1.1954	1.1928	1.1916
$p = 0.8$	K	37	33	32	30	32
	$E[A^s]$	779.2654	382.3473	334.1707	316.8086	308.4506
	K	19	18	28	15	20
	$C_v(A^s)$	1.1371	1.1016	1.0948	1.0922	1.0911

Table 4
Correlation coefficient between R^s and R^b.

		$s = 0.1$	$s = 0.3$	$s = 0.5$	$s = 0.7$	$s = 0.9$
$p = 0.2$	K	6	6	6	6	6
	$\rho(R^s, R^b)$	0.3710	0.5292	0.5907	0.6225	0.6413
$p = 0.4$	K	9	9	10	10	11
	$\rho(R^s, R^b)$	0.5939	0.6964	0.7246	0.7355	0.7401
$p = 0.6$	K	15	15	19	16	16
	$\rho(R^s, R^b)$	0.8400	0.8320	0.8264	0.8212	0.8173
$p = 0.8$	K	31	33	30	34	31
	$\rho(R^s, R^b)$	0.9751	0.9401	0.9241	0.9154	0.9110

Table 5
Correlation coefficient between R^s and A^s.

		$s = 0.1$	$s = 0.3$	$s = 0.5$	$s = 0.7$	$s = 0.9$
$p = 0.2$	K	6	6	6	6	6
	$\rho(R^s, A^s)$	0.1217	0.1064	0.1036	0.1023	0.1016
$p = 0.4$	K	8	7	8	8	7
	$\rho(R^s, A^s)$	0.4672	0.4065	0.3937	0.3881	0.3850
$p = 0.6$	K	14	17	12	12	13
	$\rho(R^s, A^s)$	0.8345	0.7752	0.7611	0.7552	0.7522
$p = 0.8$	K	22	19	19	24	19
	$\rho(R^s, A^s)$	0.9853	0.9687	0.9639	0.9619	0.9611

Table 6
Correlation coefficient between R^b and A^s.

		$s=0.1$	$s=0.3$	$s=0.5$	$s=0.7$	$s=0.9$
$p=0.2$	K	6	7	6	6	6
	$\rho(R^b, A^s)$	0.0394	0.0468	0.0498	0.0510	0.0516
$p=0.4$	K	9	9	9	9	10
	$\rho(R^b, A^s)$	0.2277	0.2173	0.2142	0.2116	0.2094
$p=0.6$	K	16	18	19	18	16
	$\rho(R^b, A^s)$	0.6092	0.5294	0.5077	0.4964	0.4899
$p=0.8$	K	30	30	30	33	32
	$\rho(R^b, A^s)$	0.9364	0.8621	0.8351	0.8220	0.8156

Table 7
Probability distribution of R^s: $P\{R^s = r\}$, $r \geq 0$.

	r	$s=0.1$	$s=0.3$	$s=0.5$	$s=0.7$	$s=0.9$
$p=0.2$	0	0.998749	0.998749	0.998749	0.998749	0.998749
$p=0.4$	0	0.939572	0.939572	0.939572	0.939572	0.939572
	1	0.037921	0.037378	0.036742	0.036249	0.035858
	2	0.013352	0.013357	0.013417	0.013476	0.013529
	3			0.005566	0.005710	0.005825
$p=0.6$	0	0.560533	0.560533	0.560533	0.560533	0.560533
	1	0.091047	0.096865	0.096441	0.095527	0.094635
	2	0.063737	0.068386	0.068294	0.067848	0.067403
	⋮	⋮	⋮	⋮	⋮	⋮
	20	0.002466	0.001808	0.001798	0.001834	0.001871
	21	0.002097				
	22	0.001784				
	23	0.001517				
$p=0.8$	0	0.150061	0.150061	0.150061	0.150061	0.150061
	1	0.015364	0.019389	0.020123	0.020291	0.020292
	2	0.011117	0.015636	0.016549	0.016822	0.016900
	⋮	⋮	⋮	⋮	⋮	⋮
	300	0.000774	0.000245	0.000176	0.000154	0.000146
	⋮	⋮	⋮	⋮	⋮	
	305	0.000750	0.000231	0.000164	0.000144	
	⋮	⋮	⋮	⋮		
	317	0.000696	0.000199	0.000139		
	⋮	⋮	⋮			
	355	0.000548	0.000124			
	⋮	⋮				
	700	0.000062				

The behavior of $E[R^s]$, $E\left[R^b\right]$, $E[A^s]$, $C_v(R^s)$, $C_v(R^b)$ and $C_v(A^s)$ as functions of s is more complex. From Table 1 we can observe that $E[R^s]$ is an increasing function of s when $p \in \{0.2, 0.4\}$, it presents a minimum when $p = 0.6$, and it is a decreasing function of s when $p = 0.8$. For $p \in \{0.2, 0.4\}$ $C_v(R^s)$ increases as far as s becomes larger, whereas for $p \in \{0.6, 0.8\}$ it decreases as a function of s. $E\left[R^b\right]$ is an increasing function of s when $p \in \{0.2, 0.4, 0.6\}$ and it reaches a minimum when $p = 0.8$. For $p = 0.2$ $C_v(R^b)$ decreases with s, for $p = 0.4$ there is a minimum and for $p \in \{0.6, 0.8\}$ it increases with increasing values of s, such as we can observe in Table 2. A possible explanation of this behavior can be the following reasoning. The length of the busy period decreases when the retrial probability s increases and, as a result, the number of events during the busy period also diminishes. On the other hand, the blocking probability (i.e., the probability of finding all the servers busy) is an increasing function of s. These two opposite phenomena explain the unexpected behavior of the moments of R^s and R^b. Finally, as was expected, Table 3 reveals that the expectation and the coefficient of variation of A^s are decreasing functions of s for each fixed p.

The correlation coefficients $\rho(R^s, R^b)$, $\rho(R^s, A^s)$ and $\rho(R^b, A^s)$ are presented in Tables 4–6, respectively. The threshold K is again chosen as the first integer matching the first four decimal digits of the corresponding correlation coefficient. As previously, the truncation levels increase with increasing values of p.

In the light of the results in these tables, we can conclude that the three coefficients present a positive correlation, no matter what values p and s take. For each fixed s, the correlation coefficients are increasing functions of p, i.e., if the traffic

Table 8
Probability distribution of R^b: $P\{R^b = r\}$, $r \geq 0$.

	r	$s = 0.1$	$s = 0.3$	$s = 0.5$	$s = 0.7$	$s = 0.9$
$p = 0.2$	0	0.999887	0.999734	0.999629	0.999551	0.999489
$p = 0.4$	0	0.987946	0.978534	0.973175	0.969557	0.966907
	1	0.008580	0.011774	0.012534	0.012626	0.012437
	2			0.005542	0.006081	0.006374
	3				0.003445	0.003775
	4					0.002422
$p = 0.6$	0	0.746172	0.699202	0.674058	0.658464	0.647753
	1	0.080753	0.072014	0.064271	0.058407	0.053707
	⋮	⋮	⋮	⋮	⋮	⋮
	14	0.001922	0.004065	0.005323	0.006005	0.006371
	⋮		⋮	⋮	⋮	⋮
	28		0.000821	0.001469	0.001972	0.002348
	⋮			⋮	⋮	⋮
	42			0.000545	0.000846	0.001110
	⋮				⋮	⋮
	57				0.000396	0.000568
	⋮					⋮
	73					0.000305
$p = 0.8$	0	0.181310	0.179264	0.174811	0.171624	0.169329
	1	0.016635	0.018406	0.016394	0.014691	0.013336
	⋮	⋮	⋮	⋮	⋮	⋮
	847	0.000050	0.000048	0.000097	0.000148	0.000190
	⋮		⋮	⋮	⋮	⋮
	855		0.000046	0.000094	0.000144	0.000186
	⋮			⋮	⋮	⋮
	1158			0.000033	0.000063	0.000093
	⋮				⋮	⋮
	1510				0.000024	0.000043
	⋮					⋮
	1889					0.000019

is heavy, then the correlation becomes stronger. As happened in the continuous-time equivalent, the retrial rate s does not influence the ergodicity condition $p < cq = 5 \cdot 0.2 = 1$, and consequently the traffic is heavier as p tends to one. On the other hand, the orbit changes into a waiting line with random service order when s approaches one; whereas the system is more congested when s is close to zero, since the probability that a repeated customer accesses the server is s at every slot boundary. That is why, the traffic is heavier in our examples when $p = 0.8$ and $s = 0.1$, and hence the correlation is close to one when $p = 0.8$ and $s = 0.1$.

The behavior of the correlation coefficients as functions of s gets complicated: $\rho(R^s, R^b)$ increases with s when $p \in \{0.2, 0.4\}$ and it decreases when $p \in \{0.6, 0.8\}$ (see Table 4), $\rho(R^s, A^s)$ is a decreasing function of s for each fixed p (see Table 5) and $\rho(R^b, A^s)$ is an increasing function of s for $p = 0.2$ and it is a decreasing one for $p \in \{0.4, 0.6, 0.8\}$ (see Table 6).

Looking at the Tables 4–6, we can also observe that $\rho(R^b, A^s) < \rho(R^s, A^s) < \rho(R^s, R^b)$ for each fixed pair (p, s) with $p \in \{0.2, 0.4, 0.6\}$, whereas $\rho(R^b, A^s) < \rho(R^s, R^b) < \rho(R^s, A^s)$ for each fixed pair (p, s) with $p = 0.8$.

In Tables 7–9 we present the probability mass functions of R^s, R^b and A^s, respectively, until the 99th percentile. For the probability distributions of R^b and A^s, we take the truncation level $K = 100$, it is large enough to obtain good approximations. We continue assuming that $c = 5$, $q = 0.2$, $p \in \{0.2, 0.4, 0.6, 0.8\}$ and $s \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. First of all, we observe that the probability mass functions of R^s, R^b and A^s appear to be decreasing and consequently have a single mode at 0, for every value of the pair (p, s). We also observe that the greater p is the more sparse the distributions of R^s, R^b and A^s are.

In Table 7 we can confirm that $P\{R^s = 0\}$ does not depend on retrial probability s and it decreases with p. For $p = 0.4$ the tail of the distribution of R^s is heavier when s is greater, whereas for $p \in \{0.6, 0.8\}$ the tail is heavier when s is lower.

Table 9
Probability distribution of A^s: $P\{A^s = a\}$, $a \geq 0$.

	a	$s = 0.1$	$s = 0.3$	$s = 0.5$	$s = 0.7$	$s = 0.9$
$p = 0.2$	0	0.444444	0.444444	0.444444	0.444444	0.444444
	1	0.182149	0.182149	0.182149	0.182149	0.182149
	⋮	⋮	⋮	⋮	⋮	⋮
	13	0.002983	0.002973	0.002970	0.002969	0.002969
$p = 0.4$	0	0.230769	0.230769	0.230769	0.230769	0.230769
	1	0.103742	0.103742	0.103742	0.103742	0.103742
	⋮	⋮	⋮	⋮	⋮	⋮
	51	0.000987	0.000874	0.000857	0.000851	0.000848
	52	0.000914	0.000805			
	53	0.000846				
	54	0.000783				
$p = 0.6$	0	0.117647	0.117647	0.117647	0.117647	0.117647
	1	0.050228	0.050228	0.050228	0.050228	0.050228
	⋮	⋮	⋮	⋮	⋮	⋮
	216	0.000412	0.000242	0.000217	0.000207	0.000202
	217	0.000406	0.000238	0.000212	0.000203	
	⋮	⋮	⋮	⋮		
	220	0.000388	0.000224	0.000200		
	⋮	⋮	⋮			
	228	0.000342	0.000192			
	⋮	⋮				
	280	0.000153				
$p = 0.8$	0	0.047619	0.047619	0.047619	0.047619	0.047619
	1	0.018723	0.018723	0.018723	0.018723	0.018723
	⋮	⋮	⋮	⋮	⋮	⋮
	1526	0.000176	0.000058	0.000038	0.000032	0.000029
	⋮	⋮	⋮	⋮	⋮	
	1568	0.000168	0.000052	0.000034	0.000028	
	⋮	⋮	⋮	⋮		
	1658	0.000152	0.000042	0.000027		
	⋮	⋮	⋮			
	1907	0.000115	0.000023			
	⋮	⋮				
	3995	0.000011				

The probability distribution of R^b is more sparse when s increases, for each fixed probability p, such as we can see in the Table 8. We can also observe that $P\{R^b = 0\}$ is a decreasing function of s for each fixed p, and it is a decreasing function of p for each fixed s.

Table 9 shows how the tail of the probability mass function of A^s is heavier when the probability s is lower, for each fixed p. We notice that $P\{A^s = 0\}$ does not depend on s. This is obvious, because the event $(A^s = 0)$ corresponds to the case where the busy period consists only on the service time of the initial customer, so that $P\{A^s = 0\} = q\bar{p}/(1 - \bar{q}\bar{p})$.

Finally, we introduce a simple optimization problem related to the mean values of the descriptors under study to illustrate one of their many applications.

Every customer (primary or repeated) that is blocked has a cost associated. It is interesting to plan how the total cost can be minimized. Obviously, different cost functions can be considered depending on the nature of the problem. We refer to the optimization problem used in the $M/M/c$ retrial queue, the continuous-time counterpart of the $Geo/Geo/c$ retrial queue, [15].

The problem deals with finding the retrial probability s for which the total expected cost is minimized. That is,

$$\min_s \left[E[R^b] + c_r E[A^b] \right],$$

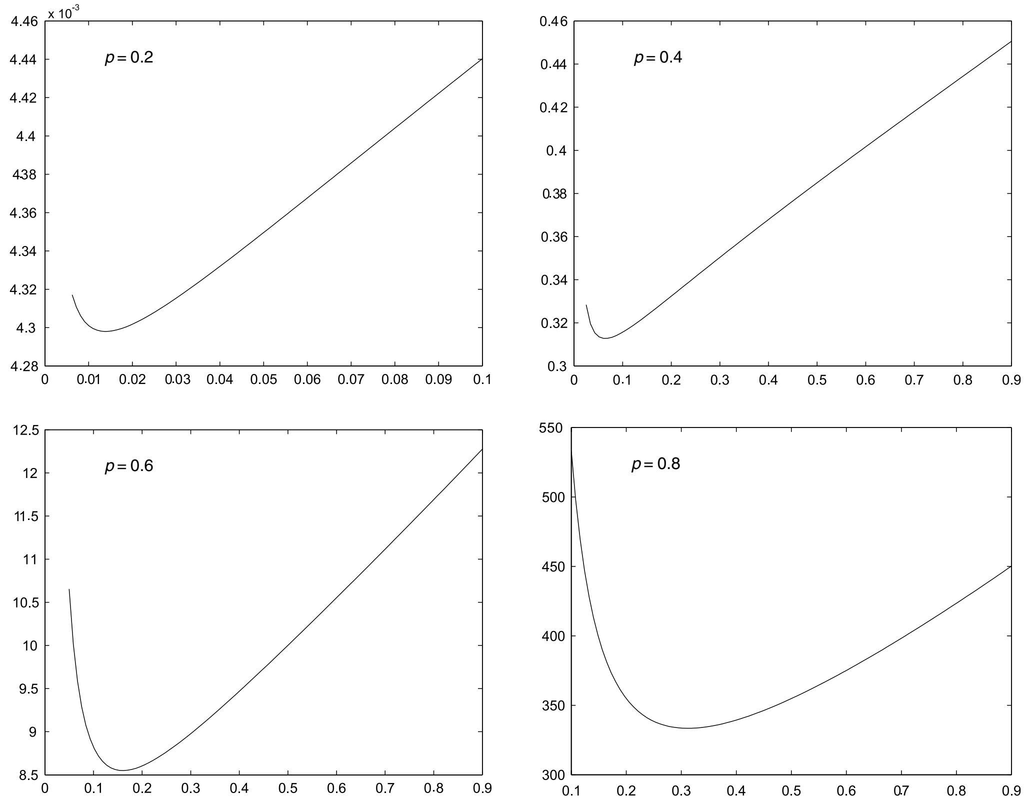

Fig. 1. $E[R^b] + c_r E[R^s]$ versus s.

where $c_r = c(A^b)/c(R^b)$, being $c(A^b)$ the unit cost per each primary arrival that is blocked and $c(R^b)$ the unit cost per each time that a retrial is blocked. Since $E[A^b] = E[R^s]$, the problem reduces to

$$\min_s \left[E[R^b] + c_r E[R^s]\right].$$

Fig. 1 illustrates results of the total expected cost, $E[R^b]+c_r E[R^s]$, versus s, for $c = 5$, $q = 0.2$ and $p \in \{0.2, 0.4, 0.6, 0.8\}$. We fix the relative cost as $c_r = 3$ and we take the truncation level $K = 100$. We observe that for $p = 0.2$ the cost function reaches a minimum at $s = 0.0138$, being the minimum value equal to 0.0042. For $p = 0.4$ the minimum is reached at $s = 0.0642$ and the minimum total expected cost is 0.3127. For $p = 0.6$ and $p = 0.8$ the minimum total expected costs are 8.5493 and 333.4357, and they are reached at $s = 0.1601$ and $s = 0.3120$, respectively. We observe that the values s which minimize the cost function increase with p and the minimum total costs also increase with p.

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